

A REMARK ON THE MEAN-FIELD DYNAMICS OF MANY-BODY BOSONIC SYSTEMS WITH RANDOM INTERACTIONS

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ABSTRACT. The mean-field limit for the dynamics of bosons with random interactions is rigorously studied. It is shown that, for interactions that are almost surely bounded, the many-body quantum evolution can be replaced in the mean-field limit by a single particle nonlinear evolution that is described by the Hartree equation. This is an Egorov-type theorem for many-body quantum systems with random interactions.

1. INTRODUCTION

This work is a modest contribution to the mathematical theory of the mean-field limit for bosons with random interactions. There has been substantial developments in the study of the mean-field dynamics of bosons with deterministic interactions. Early results were proven by Hepp in [1], see also [2]. A different approach based on the reduced density matrix was developed in [3] and was substantially extended to more general potentials and to the derivation of the Gross-Pitaevskii equation in [4], [5], [6], [7], [8]. Recently, a new approach was developed in [9], which gives convergence estimates in the mean-field limit that are uniform in Planck's constant \hbar , see also [10].

While the mean-field dynamics for bosons with deterministic interactions has attracted considerable interest, the question of the mean-field dynamics of bosons with random interactions has not been addressed, yet. Many-body bosonic systems with random interactions are relevant to concrete physical systems, such as inhomogeneous nonlinear optical media, or Bose-Einstein experiments where irregular fluctuations in currents inside conductors close to the condensate induce via Feshbach resonances inhomogeneous interactions between the bosons, see [11] for a description of the latter; also [12] and references therein. Here, we give a simple recipe for extending the deterministic mean-field analysis to the case of random interactions (and in the presence of a random potential).

1.1. The model. Consider the probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, such that the probability space Ω has a generic point ω and is endowed with measure μ . Define on this space the random field

$$v(x, \omega) : \mathbb{R}^3 \times \Omega \rightarrow \mathbb{R},$$

such that v is measurable in $x \in \mathbb{R}^3$ and $\omega \in \Omega$, and is almost surely in $L^\infty(\mathbb{R}^3)$, i.e. there exists $\Omega_0 \subset \Omega$ such that $\mu(\Omega_0) = 1$ and, for all $\omega \in \Omega_0$, $v(\cdot, \omega) \in L^\infty(\mathbb{R}^3)$. A concrete example of v that satisfies the above conditions is $v(x, \omega) = v_1(x) + v_2(x, \omega)$, such that $v_1 \in L^\infty$ and v_2 is Gaussian with finite mean and variance. For a measurable and integrable function f on Ω , we define the expectation value of f as

$$\mathbb{E}(f) := \int f(\omega) \mu(d\omega).$$

We consider the N -body random Schrödinger operator

$$(1) \quad H^N \equiv H_\omega^N := - \sum_{i=1}^N \Delta_i + \frac{1}{N} \sum_{1 \leq i < j \leq N} v(x_i - x_j, \omega),$$

where $\Delta = \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2}$ is the 3-dimensional Laplacian and $\omega \in \Omega$. Here, we work in units where Planck's constant $\hbar = 1$ and the mass of each particle is $m = \frac{1}{2}$. We note that the analysis below is uniform in \hbar . The Hamiltonian H^N acts on the Hilbert space $\mathcal{H}^N := L_S^2(\mathbb{R}^{3N})$, the symmetrization of $L^2(\mathbb{R}^{3N})$, which is the space of pure states for a system of N nonrelativistic bosons.

The quantum dynamics of the N -body system is described by the Schrödinger equation

$$(2) \quad i\partial_t \Psi^N(t) = H^N \Psi^N(t),$$

with an initial condition $\Psi^N(t=0) = \Psi^{N,0} \in L_S^2(\mathbb{R}^{3N})$.

Together with the dynamics defined above, the N -body system is described by a kinematical algebra of “observables”. For $p \leq N$, a p -particle observable is described by an operator $a^{(p)} \in \mathcal{B}(\mathcal{H}^{(p)})$, where $\mathcal{B}(\mathcal{H}^{(p)})$ is the algebra of bounded operators on $\mathcal{H}^{(p)} = L_S^2(\mathbb{R}^{3p})$. By the nuclear theorem, one can associate with $a^{(p)}$ a tempered distribution kernel in $\mathcal{S}'(\mathbb{R}^{3p} \times \mathbb{R}^{3p})$, $\alpha^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p) := \alpha^{(p)}(X_p; Y_p)$, such that

$$(3) \quad (a^{(p)} \varphi^{(p)})(X_p) = \int_{\mathbb{R}^{3p}} \alpha^{(p)}(X_p; Y_p) \varphi^{(p)}(Y_p) dY_p$$

where $\varphi^{(p)}(Y_p) \in L_S^2(\mathbb{R}^{3p})$. We associate to $a^{(p)}$ an operator $A^N(a^{(p)})$ acting on $\mathcal{H}^{(N)}$ that is given by

$$(4) \quad (A^N(a^{(p)})\Psi)(x_1, \dots, x_N) = \frac{N!}{N^p(N-p)!} (P_S a^{(p)} \otimes I^{(N-p)} P_S \Psi)(x_1, \dots, x_N),$$

where $\Psi(x_1, \dots, x_N) \in L_S^2(\mathbb{R}^{3N})$ and P_S is the projection onto the symmetric subspace $L_S^2(\mathbb{R}^{3N})$ of $L^2(\mathbb{R}^{3N})$. It follows from (3) and (4) that the map

$$A^N : \mathcal{B}(\mathcal{H}^{(p)}) \rightarrow \mathcal{B}(\mathcal{H}^{(N)}), \quad 1 \leq p \leq N,$$

is linear, such that

$$(5) \quad \begin{aligned} \|A^N(a^{(p)})\|_{\mathcal{B}(\mathcal{H}^{(N)})} &\leq \|a^{(p)}\|_{\mathcal{B}(\mathcal{H}^{(p)})}, \\ A^N(a^{(p)})^* &= A^N(a^{(p)*}). \end{aligned}$$

In the Heisenberg picture, the evolution of $A^{(N)} \in \mathcal{B}(\mathcal{H}^{(N)})$ is given by

$$(6) \quad \alpha_t^N(A^{(N)}) := e^{iH^N t} A^{(N)} e^{-iH^N t}, \quad t \in \mathbb{R}.$$

Since v is almost surely bounded, H^N is almost surely self-adjoint on the symmetrized Sobolev space $H_S^2(\mathbb{R}^{3N})$, and hence the propagator $e^{-iH^N t}$, $t \in \mathbb{R}$, is almost surely unitary. Moreover, it follows from the fact that the pointwise limit of measurable functions is itself measurable, [13], and the Trotter product formula, [14], that

$$\langle \otimes_{j=1}^N \psi_j(x_j), \alpha_t^N(A^{(N)}) \otimes_{j=1}^N \psi_j(x_j) \rangle, \quad A^{(N)} \in \mathcal{B}(\mathcal{H}^{(N)}), \quad \psi_j \in L^2(\mathbb{R}^3)$$

is ω -measurable.

We now introduce the *classical* evolution. The Hartree equation is given by

$$(7) \quad i\partial_t \psi_t = -\Delta \psi_t + (v \star |\psi_t|^2) \psi_t,$$

with the initial condition $\psi_{t=0} = \phi \in L^2(\mathbb{R}^3)$. It follows from Duhamel's formula for ψ_t and the fact that $v \in L^\infty$ almost surely, that global solutions of (7) in L^2 exist almost surely, such that $\|\psi_t\|_{L^2} = \|\phi\|_{L^2}$ with probability 1, for all $t \in \mathbb{R}$, (see for example [15] for the case when $v \in L^\infty$). It also follows from Duhamel's formula that the random variable

$$\langle \otimes_{i=1}^p \psi_t, A^{(p)} \otimes_{i=1}^p \psi_t \rangle, \quad A^{(p)} \in \mathcal{B}(\mathcal{H}^{(p)})$$

is ω -measurable.

1.2. Statement of the main result. We are in a position to state the main result.

Theorem 1. *Given $a^{(p)}$, $A^N(a^{(p)})$ and α_t^N as above, suppose that the initial state of the N -body system is a normalized coherent (product) state $\Psi^{N,0}(x_1, \dots, x_N) = \otimes_{i=1}^N \phi(x_i)$, $\phi \in L^2(\mathbb{R}^3)$. Then, for fixed $t \geq 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{E}(\langle \Psi^{N,0}, \alpha_t^N(A^N(a^{(p)})) \Psi^{N,0} \rangle) = \mathbb{E}(\langle \otimes_{i=1}^p \psi_t, A^{(p)} \otimes_{i=1}^p \psi_t \rangle),$$

where ψ_t satisfies the Hartree equation (7) with initial condition $\psi_{t=0} = \phi$.

We note that the analysis below can be easily extended to study the mean-field dynamics of bosons in a random external potential that is almost surely smooth, polynomially bounded and positive, and to investigate the semi-classical limit of the dynamics under additional assumptions on the decay of the interaction, as in [9]. Furthermore, the analysis below can be applied “in toto” to extend the results of [10] and [16] to the case of random interactions.

2. PROOF OF THEOREM 1

The proof of Theorem 1 follows effectively from an application of the dominated convergence theorem, see [13], and Theorem 1.1 in [9]. In what follows, we drop the explicit dependence on the time t in the notation, since we fix it.

Proof. We introduce the random variables

$$X_N^{(p)} := \langle \Psi^{N,0}, \alpha_t^N(A^N(a^{(p)})) \Psi^{N,0} \rangle$$

and

$$X^{(p)} := \langle \otimes_{i=1}^p \psi_t, a^{(p)} \otimes_{i=1}^p \psi_t \rangle.$$

The claim of the theorem is equivalent to the statement

$$(8) \quad \lim_{N \rightarrow \infty} \mathbb{E}(X_N^{(p)}) = \mathbb{E}(X^{(p)}).$$

We divide the proof of (8) into several steps.

Step 1. Uniform integrability. We want to show that

$$(9) \quad \lim_{\beta \rightarrow \infty} \mathbb{E}(|X_N^{(p)}| \mathbf{1}_{|X_N^{(p)}| \geq \beta}) = 0,$$

uniformly in $N \in \mathbb{N}$.

We have from (5) and the fact that the quantum time-evolution is almost surely unitary, that

$$(10) \quad |X_N^{(p)}| \leq \|a^{(p)}\|_{\mathcal{B}(\mathcal{H}^{(p)})} < 2\|a^{(p)}\|_{\mathcal{B}(\mathcal{H}^{(p)})} < \infty, \text{ almost surely,}$$

uniformly in $N \in \mathbb{N}$. For $\beta > 0$, it follows from (10) that

$$(11) \quad |X_N^{(p)}| \mathbf{1}_{|X_N^{(p)}| \geq \beta} \leq |X_N^{(p)}| < 2\|a^{(p)}\|_{\mathcal{B}(\mathcal{H}^{(p)})} < \infty, \text{ almost surely,}$$

uniformly in $N \in \mathbb{N}$. The dominated convergence theorem together with (11) give (9).

Step 2. Mean-field limit with probability 1. It follows from the fact that the particle interaction $v \in L^\infty$ almost surely and Theorem 1.1 in [9] that, for fixed $t > 0$,

$$(12) \quad X_N^{(p)} \xrightarrow{N \rightarrow \infty} X^{(p)} \text{ almost surely.}$$

Step 3. It follows from Fatou's lemma, [13], and (10), that

$$(13) \quad \mathbb{E}(|X^{(p)}|) \leq \liminf_N \mathbb{E}(|X_N^{(p)}|) \leq \limsup_N \mathbb{E}(|X_N^{(p)}|) < 2\|a^{(p)}\|_{\mathcal{B}(\mathcal{H}^{(p)})} < \infty,$$

uniformly in $N \in \mathbb{N}$. We also have that

$$|X^{(p)}| \mathbf{1}_{|X^{(p)}| \geq \beta} \leq |X^{(p)}|,$$

which together with (13) and the dominated convergence theorem, imply that

$$(14) \quad \lim_{\beta \rightarrow \infty} \mathbb{E}(|X^{(p)}| \mathbf{1}_{|X^{(p)}| \geq \beta}) = 0.$$

Step 4. Convergence as $N \rightarrow \infty$. We introduce the random variable

$$Y_N^{(p)} := |X^{(p)} - X_N^{(p)}|.$$

Note that it suffices to show that $\mathbb{E}(Y_N^{(p)}) \rightarrow 0$ as $N \rightarrow \infty$, from which (8) follows by the triangular inequality.

It follows from (12), Step 2, that

$$(15) \quad Y_N^{(p)} \xrightarrow{N \rightarrow \infty} 0 \text{ almost surely.}$$

We decompose $Y_N^{(p)}$ into two parts,

$$Y_N^{(p)} = Y_N^{(p), < \beta} + Y_N^{(p), \geq \beta},$$

where $Y_N^{(p), < \beta} := Y_N^{(p)} \mathbf{1}_{|Y_N^{(p)}| < \beta}$ and $Y_N^{(p), \geq \beta} := Y_N^{(p)} \mathbf{1}_{|Y_N^{(p)}| \geq \beta}$, for $\beta > 0$.

Since $Y_N^{(p), < \beta} < \beta$, (15) together with the dominated convergence theorem imply that

$$(16) \quad \lim_{N \rightarrow \infty} \mathbb{E}(Y_N^{(p), < \beta}) = 0.$$

Furthermore, since

$$Y_N^{(p), \geq \beta} \leq 2|X^{(p)}| \mathbf{1}_{|X^{(p)}| \geq \beta/2} + 2|X_N^{(p)}| \mathbf{1}_{|X_N^{(p)}| \geq \beta/2},$$

it follows from (9) and (14) that

$$(17) \quad \lim_{\beta \rightarrow \infty} \mathbb{E}(Y_N^{(p), \geq \beta}) = 0,$$

uniformly in $N \in \mathbb{N}$.

Given $\epsilon > 0$, (17) implies that there exists a finite $\beta_0 > 0$ such that

$$\sup_N \mathbb{E}(Y_N^{(p), \geq \beta_0}) < \epsilon/2.$$

Moreover, (16) implies that there exists a positive integer N_0 such that, for all $N \geq N_0$,

$$\mathbb{E}(Y_N^{(p), < \beta_0}) < \epsilon/2.$$

It follows that

$$\mathbb{E}(Y_N^{(p)}) = \mathbb{E}(Y_N^{(p), < \beta_0}) + \mathbb{E}(Y_N^{(p), \geq \beta_0}) < \epsilon$$

for $N \geq N_0$. Therefore, $\mathbb{E}(Y_N^{(p)}) \xrightarrow{N \rightarrow \infty} 0$.

By the triangular inequality,

$$|\mathbb{E}(X^{(p)}) - \mathbb{E}(X_N^{(p)})| \leq \mathbb{E}(Y_N^{(p)}) \xrightarrow{N \rightarrow \infty} 0,$$

which gives the claim of the theorem. \square

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